Generalized Reverse Rearrangement

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Abstract

My name is Daniel Liu, and in this paper I will be revisiting the Reverse Rearrangement Inequality [1] that I discovered in late 2014. I will first supply an alternate proof of the familiar two-sequence RR Inequality, then prove the general inequality for n sequences.

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Contents

1 Definitions and Notation

Two *similarly ordered* sequences $\{a_k\}$ and $\{b_k\}$ satisfy that a_n is the mth biggest term in the sequence $\{a_k\}$ iff b_n is the mth biggest term in the sequence $\{b_k\}$.

The sequence $\sigma = \{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ is defined as an arbitrary permutation of the sequence $\{1, 2, \ldots, n\}$. Thus, $\sigma_1, \sigma_2, \ldots, \sigma_n$ are n (not necessarily distinct) permutations of $\{1, 2, \ldots, n\}$.
The identity permutation I satisies that $I(k) = k$ for all $1 \leq k \leq n$.
σ' is the inverse permutation of σ, defined as $\sigma'(\sigma(k)) = k$ for all $1 \leq k \leq n$.

2 Reverse Rearrangement Inequality

We shall begin by stating the definition of the Reverse Rearrangement Inequality.

Reverse Rearrangement Inequality (RR)

Given two similarly ordered positive real sequences $\{a_k\}$ and $\{b_k\}$, the double inequality

$$\prod_{k=1}^{n}(a_k + b_k) \leq \prod_{k=1}^{n}(a_k + b_{\sigma(k)}) \leq \prod_{k=1}^{n}(a_k + b_{n-k+1})$$

is true.

The proof is found in [1]. Here I present an alternate proof.

Proof. WLOG $\{a_k\}$ and $\{b_k\}$ are non-decreasing sequences.

We want to find the minimum value of

$$\prod_{k=1}^{n}(a_k + b_{\sigma(k)})$$

If $\sigma = I$, then obviously we have achieved the desired minimum of

$$\prod_{k=1}^{n}(a_k + b_{\sigma(k)}) \geq \prod_{k=1}^{n}(a_k + b_k)$$

Otherwise, there exists an integer $1 \leq i \leq n$ such that

$$i < \sigma(i) \text{ and } \sigma(i) > \sigma(\sigma(i))$$

because if there didn't, then either (for all $1 \leq k \leq n$) $\sigma(k) = k$ (which we have already discussed), $\sigma(k) > k$, or $\sigma(k) < k$, the latter two of which are clearly impossible for finite n.

Now note that

$$(a_i + b_{\sigma(i)})(a_{\sigma(i)} + b_{\sigma(\sigma(i))}) \geq (a_i + b_{\sigma(\sigma(i))})(a_{\sigma(i)} + b_{\sigma(i)})$$

because the above inequality expands and rearranges to

$$(a_{\sigma(i)} - a_i)(b_{\sigma(i)} - b_{\sigma(\sigma(i))}) \geq 0$$

which is true since

$$\sigma(i) > i \implies a_{\sigma(i)} \geq a_i$$
$$\sigma(i) > \sigma(\sigma(i)) \implies b_{\sigma(i)} \geq b_{\sigma(\sigma(i))}$$

which follows because $\{a_k\}, \{b_k\}$ are non-decreasing sequences.

Define the permutation σ_0 to be identical with σ except $\sigma_0(i) = \sigma(\sigma(i))$ and $\sigma_0(\sigma(i)) = \sigma(i)$.

With this, we arrive at the conclusion that

$$\begin{aligned}\prod_{k=1}^{n}(a_k + b_{\sigma(k)}) &= (a_i + b_{\sigma(i)})(a_{\sigma(i)} + b_{\sigma(\sigma(i))}) \prod_{1 \leq k \neq i, \sigma(i) \leq n} (a_k + b_{\sigma(k)}) \\ &\geq (a_i + b_{\sigma(\sigma(i))})(a_{\sigma(i)} + b_{\sigma(i)}) \prod_{1 \leq k \neq i, \sigma(i) \leq n} (a_k + b_{\sigma(k)}) \\ &= \prod_{k=1}^{n}(a_k + b_{\sigma_0(k)})\end{aligned}$$

But clearly this process can be repeated as long as $\sigma \neq I$, so if it terminates, it will always terminate at

$$\prod_{k=1}^{n}(a_k + b_{\sigma(k)}) \geq \prod_{k=1}^{n}(a_k + b_k)$$

So we only have to prove that this process terminates.

But this is easy: there are only a finite number of permutations of $\{1, 2, \ldots, n\}$ possible, and it is impossible to end up with the same permutation twice with this process. This can be seen through a proof by contradiction:

Let P be the product corresponding to the permutation that appears twice in the process. Then we know that

$$P \geq \cdots \geq P$$

However, letting $a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots b_n$ all be distinct real numbers removes the equality case, giving us

$$P > \cdots > P$$

contradiction. $\square$

3 Generalized Reverse Rearrangement Inequality

This inequality can also be generalized to n sequences:

Generalized RR

Given n similarly ordered positive real sequences $\{a_{i,j}\}_{1\le i\le m}$ for $1 \le j \le n$, the inequality

$$\prod_{i=1}^{m}\sum_{j=1}^{n} a_{\sigma_j(i),j} \ge \prod_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}$$

is true.

Proof. WLOG let the n sequences $\{a_i^j\}$ for $1 \le j \le n$ all be non-decreasing.

We want to find the minimum possible value of

$$\prod_{i=1}^{m}\sum_{j=1}^{n} a_{\sigma_j(i),j}$$

Clearly, if $\sigma_1 = \sigma_2 = \cdots = \sigma_n$, then the minimum value

$$\prod_{i=1}^{m}\sum_{j=1}^{n} a_{\sigma_j(i),j} \ge \prod_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}$$

is achieved as intended.

Otherwise, denote

$$x_{\sigma(i)} = \sum_{j=1}^{n} a_{\sigma_j(i),j}$$

where σ is an arbitrary permutation that satisfies $x_1 \le x_2 \le \cdots \le x_m$. Then, defining $\sigma'_j(i) = \sigma_j(\sigma'(i))$, we get

$$x_i = \sum_{j=1}^{n} a_{\sigma'_j(i),j}$$

By inspection

$$\sigma_1 = \sigma_2 = \cdots = \sigma_n \iff \sigma'_1 = \sigma'_2 = \cdots = \sigma'_n = I$$

I claim that if $\sigma'_1 = \sigma'_2 = \cdots = \sigma'_n = I$ is not true, then there exists $1 \le p \le m$, $1 \le q \le n$ that satisfies

$$p < \sigma'_q(p) \text{ and } \sigma'_q(p) > \sigma'_q(\sigma'_q(p))$$

The proof of this is easy: take q to be an integer that satisfies $\sigma'_q \neq I$, then in order for the statement to be false we must either have, for all $1 \leq p \leq m$, $\sigma'_q(p) = p$, $\sigma'_q(p) > p$, or $\sigma'_q(p) < p$, all of which are impossible.

Now note that

$$x_p x_{\sigma'_q(p)} \geq (x_p - a_{\sigma'_q(p),q} + a_{\sigma'_q(\sigma'_q(p)),q})(x_{\sigma'_q(p)} - a_{\sigma'_q(\sigma'_q(p)),q} + a_{\sigma'_q(p),q})$$

because it rearranges to

$$(a_{\sigma'_q(p),q} - a_{\sigma'_q(\sigma'_q(p)),q})(x_{\sigma'_q(p)} - x_p + a_{\sigma'_q(p),q} - a_{\sigma'_q(\sigma'_q(p)),q}) \geq 0$$

which is true since

$$p < \sigma'_q(p) \implies x_p \leq x_{\sigma'_q(p)}$$

$$\sigma'_q(p) > \sigma'_q(\sigma'_q(p)) \implies a_{\sigma'_q(p),q} \geq a_{\sigma'_q(\sigma'_q(p)),q}$$

which follows because $\{x_k\}$, $\{a_{i,j}\}$ are non-decreasing sequences.

Define the permutation $\sigma'_{j0}(i)$ to be identical to $\sigma'_j(i)$ except that $\sigma'_{q0}(p) = \sigma'_q(\sigma'_q(p))$ and $\sigma'_{q0}(\sigma'_q(p)) = \sigma'_q(p)$.
Then define the sequence $\{x_{k0}\}$ as

$$x_{i0} = \sum_{j=1}^{n} a_{\sigma'_{j0}(i),j}$$

With this, we see that

$$\begin{aligned}
\prod_{i=1}^{m} \sum_{j=1}^{n} a_{\sigma_j(i),j} &= \prod_{i=1}^{m} x_{\sigma(i)} \\
&= \prod_{i=1}^{m} x_i \\
&= x_p x_{\sigma'_q(p)} \prod_{1 \leq i \neq p, \sigma'_q(p) \leq m} x_i \\
&\geq (x_p - a_{\sigma'_q(p),q} + a_{\sigma'_q(\sigma'_q(p)),q})(x_{\sigma'_q(p)} - a_{\sigma'_q(\sigma'_q(p)),q} + a_{\sigma'_q(p),q}) \prod_{1 \leq i \neq p, \sigma'_q(p) \leq m} x_i \\
&= x_{p0} x_{\sigma'_q(p)0} \prod_{1 \leq i \neq p, \sigma'_q(p) \leq m} x_{i0} \\
&= \prod_{i=1}^{m} x_{i0} \\
&= \prod_{i=1}^{m} \sum_{j=1}^{n} a_{\sigma'_{j0}(i),j}
\end{aligned}$$

So in conclusion when $\sigma_1 = \sigma_2 = \cdots = \sigma_n$ is not true, we have

$$\prod_{i=1}^{m}\sum_{j=1}^{n} a_{\sigma_j(i),j} \geq \prod_{i=1}^{m}\sum_{j=1}^{n} a_{\sigma'_{j0}(i),j}$$

However, we can apply this inequality as long as the process does not terminate at $\sigma_1 = \sigma_2 = \cdots = \sigma_n$, so it remains to prove that the process terminates. The proof of termination is identical to the proof of termination for the two-sequence version of Reverse Rearrangement, so it must terminate to $\sigma_1 = \sigma_2 = \cdots = \sigma_n$.

Therefore,

$$\prod_{i=1}^{m}\sum_{j=1}^{n} a_{\sigma_j(i),j} \geq \prod_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}$$

is true, and we are done. □

4 The AM-GM Test

Can General RR prove AM-GM? Indeed, WLOG let $a_1 \leq a_2 \leq \cdots \leq a_n$. By General RR,

$$\begin{aligned} &(a_1 + a_2 + \cdots + a_n)(a_2 + a_3 + \cdots + a_n)\cdots(a_n + a_1 + \cdots + a_{n-1}) \\ \geq\ &(a_1 + a_1 + \cdots + a_1)\cdots(a_n + a_n + \cdots + a_n) \end{aligned}$$

which simplifies as

$$\left(\sum a_i\right)^n \geq n^n \prod a_i \iff \sum a_i \geq n\sqrt[n]{\prod a_i}$$

which is AM-GM, proved. □

References

[1] Reverse Rearrangement Inequality